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Envy-Free Divisions

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ABSTRACT. We consider the division of a single homogeneous object and transfers of money among several people who may have different valuations for the object. A division is envy free if every person believes the division he or she received is at least as valuable as the division received by each other person. If no money is transferred, the only envy-free division involves each person receiving the same fraction of the object. When money transfers are allowed, we show that the set of envy-free divisions is a simplex whose vertices involve giving the same fraction of the object to each person in a set of persons who most value the object, and in turn those people pay the same amount of money to the other people who receive none of the object.

1. Introduction

We consider the division of a single homogeneous object among several people who are willing to transfer money among themselves. A division is envy free if every person believes they received at least as much value from the object and money transfer as they believe every other person received. We characterize the set of envy-free divisions.

2. A Motivating Example

Consider the problem of three people, Ann, Ben and Carl, trying to divide up a collection of toy cars. Individual toy cars is not a single homogeneous object, but the collection might be thought of that way if, for example, the people are willing to time share some of the cars. Each person believes he or she is entitled to at least one third of the worth of the entire collection. One problem is figuring out what the collection is actually worth. Ann loves little toy cars and she has been wanting to start a collection of her own, so she believes this collection is worth \$24. Ben, who has fond memories of what it was like to play with toy cars when he was young, believes the collection is worth \$18. Finally, Carl sees the collection as something to sell, and he believes he can receive \$12 for it, so the collection is worth \$12 to Carl.

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They initially propose to divide up the car collection in such a way that Ann receives two-thirds of the collection and has to pay \$6 to the others, while Ben receives one-third of the collection and receives \$2, and Carl receives only \$4. Of course we are assuming here that it is physically possible to divide the collection into thirds, possibly via a time share. We will also assume that each person thinks a one-third portion of the collection to be worth one-third of the entire collection. With these assumptions, everyone got at least one-third of what they value the collection to be worth. For example, Ben values the division of the collection he received at $(\$18)\frac{1}{3} + \$2 = \$8$, which is the \$18 he thinks the collection is worth multiplied times the portion of the collection received, $\frac{1}{3}$, plus the money he received from Ann. So Ben thought he received $\frac{\$8}{\$18} = \frac{4}{9}$ of the collection's worth, which is more than $\frac{1}{3}$. Carl did receive his $\frac{1}{3}$ of the value of the collection, \$4 out of \$12, but he is dissatisfied for a different reason. From Carl's perspective, Ben received $(\$12) \cdot \frac{1}{3} + \$2 = \$6$, which is greater than what Carl received. Thus, Carl is envious of what Ben received.

We are interested in divisions of the collection and transfers of money in which no one is envious of anyone else. There are many such envy-free divisions. Here are five such divisions for this situation:

Division Number	Portion of Collection (Ann, Ben, Carl)	Money Given (Ann, Ben, Carl)
1	$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$(0, 0, 0)$
2	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$(2, 2, -4)$
3	$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$(3, 3, -6)$
4	$(1, 0, 0)$	$(12, -6, -6)$
5	$(1, 0, 0)$	$(16, -8, -8)$

In division 2 the portion of the collection is $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$, so Ann gets $\frac{1}{2}$ of the collection, Ben gets $\frac{1}{2}$ of the collection and Carl gets 0 of the collection and the money given is $(2, 2, -4)$, so Ann gives \$2, Ben gives \$2, and Carl receives \$4. This division of the car collection and money transfers is envy free for the following reason. Ann and Ben are not envious of each other because they both receive the same portion of the object and give the same amount of money. Ann and Ben are not envious of Carl either because Ann believes she receives $(\$24)\frac{1}{2} - \$2 = \$10$ and Ben believes he receives $(\$18)\frac{1}{2} - \$2 = \$7$, whereas Carl receives only \$4. Carl is not envious of Ann and Ben either because Carl believes they each receive $(\$12)\frac{1}{2} - \$2 = \$4$ which is the same amount as Carl receives. The other divisions are also envy free, shown the same way.

The five divisions above are found by giving the object or portion of the object to the person or people who values it most. To find another division, which may not give the object to the people who value it the most, we can take a convex combination of these five divisions. For example, we can take $\frac{1}{3}$ of division 3, and $\frac{2}{3}$ of division 4. We can add these together to obtain the portion of the collection

$$\frac{1}{3}\left(\frac{1}{2},\frac{1}{2},0\right)+\frac{2}{3}(1,0,0)=\left(\frac{5}{6},\frac{1}{6},0\right)$$

and the money given

$$\frac{1}{3}(3,3,-6)+\frac{2}{3}(12,-6,-6)=(9,-3,-6)$$

This division is also envy-free as shown in the table below. The diagonals are the values which a person believes they received, the columns are the value each person recieved according to the row for each person,, and it is envy free if they are greater than or equal to the other values in the same row

Values of	Amount received by:		
According to:	Ann	Ben	Carl
Ann	\$11	\$7	\$6
Ben	\$6	\$6	\$6
Carl	\$1	\$5	\$6

Thus everyone thinks they received at least as much as they think everyone else received.

We found one envy-free division using a convex combination of the first five divisions, and we can find another using a different convex combination. Conversely, it is possible to find all the envy-free divisions with convex combinations of the first five. Taking it a step farther we can find all the envy-free divisions for any number of people dividing up one object, using convex combinations of certain special envy-free divisions.

3. Fair Division

Now we will consider the general problem of dividing up one homogeneous object among several different people. This leads us to the following definitions.

Definition 1. *A* fair division problem *is a pair* (m, b) *where* m *is the number of people and* b *is a vector of* m *nonnegative numbers,* b_i *indicating how much monetarily person* i *thinks the single homogeneous and divisible object is worth. Let* $M = \{1, 2, \dots, m\}$ *denote the set of people.*

In the motivating problem we can see that $m = 3$, $b = (24, 18, 12)$, and $M = \{1, 2, 3\}$ which corresponds to {Ann, Ben, Carl}. We think of the object as being homogeneous and divisible and so different amounts of the object can be given to each person. We also allow persons to exchange money. This leads to the following definition.

Definition 2. *A* division *for the fair division problem* (m, b) *is the pair* (x, d) *where* $x_i \geq 0$ *is the fraction of the object received by person* $i \in M$*, and so* $\sum_{i=1}^{m} x_i = 1$*, and where* d_i *is the amount of money person* i *gives to others, and so* $\sum_{i=1}^{m} d_i = 0$.

In the motivating example of division 2, $x = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and $d = (2, 2, -4)$. We assume that all people are self interested, are risk neutral, and find the object and money are perfect substitutes. The following definition formalizes these assumptions.

Definition 3. *Given a division (x, d) for a fair division problem (m, b), the* value *person i has for person j's division is given by $b_i x_j - d_j$.*

In the last divisions described for the motivating example, the value Carl had for Ann's division was $(\$12)\frac{5}{6} - \$9 = \$1$, which matches up with the table of everyone's valuations for the other people. In order for each person to be satisfied with the division they received they need to believe they received at least as much in value as they believe every other person received. This is called an envy-free division.

Definition 4. *A division (x, d) for the fair division problem (m, b) is* envy free *if for each person k and j, the value person k has for person k's division is at least as great as the value person k has for person j's division, equivalently,*

$$b_k x_k - d_k \geq b_k x_j - d_j$$

for all k, $j \in M$.

Again, in the motivating example convex combination division is envy free because each diagonal element in the table is at least as big as the other entrees in the same row. The initially proposed division is not envy free because Carl's value of Ben's division was larger than Carl's value of his own division. Our goal is to describe the set of envy-free divisions. The set of envy-free divisions has a special geometric shape.

Definition 5. *A convex combination is a linear combination in which all the coefficients are nonnegative and sum to one. If a set of all convex combinations of $n+1$ vectors forms an n-dimensional set, then that set is called a* simplex.

In 1-dimension a simplex would consist of two points and a line connecting them. A simplex is a line segment in 1-dimension. When we move up to 2-dimensions a simplex will have 3 vertices and each vertex is connected to the other two vertices, forming a triangle. A 2-dimensional simplex is a triangle and everything contained by it. It follows that the 3-dimensional simplex would be a tetrahedron. It would have 4 vertices with each vertex connected to the other vertices and the faces of the simplex would be copies of the previous dimension's simplex, so in this case each face is a triangle. In 4-dimensions there are 5 vertices and each vertex is connected to the other 4 vertices, with each side of this 4-dimensional shape being a 3-dimensional tetrahedron. Following this pattern we obtain a sense of what a simplex is in higher dimensions.

By interchanging person indices if necessary, we can assume that the people value the object in descending order; that is, $b_1 \geq b_2 \geq \ldots \geq b_m$. As in the motivating example, it will be important to consider giving an equal amount of the object to the n people who value it the most. That division of the object is

$$y^n = (\underbrace{\frac{1}{n}, ..., \frac{1}{n}}_{n}, \underbrace{0, ..., 0}_{m-n}) \tag{1}$$

for each $n \in M$. It will also be important to consider giving an equal amount of money from each of the n people receiving some of the object to each of the other $m - n$ people not receiving the object. Such a money transfer is proportional to the division

$$e^n = (\underbrace{\frac{1}{n} - \frac{1}{m}, ..., \frac{1}{n} - \frac{1}{m}}_{n}, \underbrace{-\frac{1}{m}, ..., -\frac{1}{m}}_{m-n}) \tag{2}$$

for each $n \in M$. Note that

$$y^n - y^m = e^n \tag{3}$$

for each $n \in M$.

4. Characterization Result

We will now show the set of envy-free divisions is in fact a simplex. The vertices are the points where we divide up the object evenly among the n people who value it the most and transferring either the least amount or the most amount of money possible with the division still being envy free for the person n and $n + 1$. In the motivating example, these are the five divisions described in the table in Section 2.

Theorem 1. *Suppose (m, b) is a fair division problem with $b_1 > b_2 > ... > b_m$. The division (x, d) is envy free if and only if (x, d) is a convex combination of the divisions in*

$$D = \{(y^n, b_n e^n);\ n = 1, ..., m\} \cup \{(y^n, b_{n+1} e^n);\ n = 1, ..., m - 1\}.$$

Furthermore, each division in D is a vertex of the set of envy-free divisions and the set of envy-free divsions is the simplex determined by D.

Proof. We will show if (x, d) is a convex combination of elements of D, then (x, d) is envy-free in Section 1; show, if (x, d) is envy-free, then (x, d) is a convex combination of D in Section 2, and finally in Section 3, we will show that the set of envy-free divisions is the simplex determined by D. Taken together, this will prove the theorem.

Section 1: We will show if (x, d) is a convex combination of D then (x, d) is envy-free. This will follow from the following three claims.

Claim 1.1: $(x, d) = (y^n, b_n e^n)$ is envy-free for all $n = 1, 2, ..., m$. Indeed, the first n people are not envious of each other because they all receive the same portion of the object and give away the same amount of money. Similarly, the last $m - n$ people are not envious of each other because they all receive no part of the object and the same amount of money. Now pick person j in the first n people and person k in the last $m - n$ people, and show j is not envious of k as well as person k is not envious of person j. Person j values her division at $\frac{1}{n} b_j - \left(\frac{1}{n} - \frac{1}{m}\right) b_n$ and values the division of person k at $\frac{1}{m} b_n$. We know $\frac{1}{n}b_j \geq \frac{1}{n}b_n = \frac{1}{m}b_n + \left(\frac{1}{n} - \frac{1}{m}\right)b_n$ which then implies $\frac{1}{n}b_j - \left(\frac{1}{n} - \frac{1}{m}\right)b_n \geq \frac{1}{m}b_n$. Thus person j is not envious of person k. Person k values his division at $\frac{1}{m}b_n$ and values person j's division at $\frac{1}{n}b_k - \left(\frac{1}{n} - \frac{1}{m}\right)b_n$. Since $\frac{1}{m}b_n + \left(\frac{1}{n} - \frac{1}{m}\right)b_n = \frac{1}{n}b_n > \frac{1}{n}b_k$ which then implies $\frac{1}{m}b_n \geq \frac{1}{n}b_k - \left(\frac{1}{n} - \frac{1}{m}\right)b_n$, person k is not envious of person j. Thus no one is envious of anyone else.

Claim 1.2: $(x,d) = (y^n, b_{n+1}e^n)$ is envy-free for all $n = 1, 2, ..., m-1$. Indeed, the proof is similar to claim 1.1 by using the same reasoning as claim 1.1 replacing b_{n+1} for b_n every place it appears.

Claim 1.3: A convex combination of envy-free divisions is an envy-free division. Indeed, the set of envy-free divisions is the intersection of sets defined by linear inequalities (see Definition 4). Each linear inequality defines a closed half-space, so the intersection of all of them will be a convex space, and thus any convex combination will also be in that space [**4**].

Section 2: Suppose (x,d) is an envy-free division. We will show (x,d) is a convex combination of the divisions in D; that is, there exist α_i and β_i satisfying

$$\begin{aligned} \alpha_i &\geq 0,\ i = 1, ..., m \\ \beta_i &\geq 0,\ i = 1, ..., m-1 \end{aligned} \tag{4}$$

$$\sum_{i=1}^{m} \alpha_i + \sum_{i=1}^{m-1} \beta_i = 1 \tag{5}$$

$$(x,d) = \sum_{i=1}^{m} \alpha_i \left(y^i, e^i b_i\right) + \sum_{i=1}^{m-1} \beta_i \left(y^i, e^i b_{i+1}\right) \tag{6}$$

We will do this by letting

$$\alpha_i = i\frac{(b_{i+1}\ x_{i+1}\ -\ d_{i+1}) - (b_{i+1}\ x_i\ -\ d_i)}{b_i\ -\ b_{i+1}} \tag{7}$$

for $i = 1, ..., m-1$,

$$\beta_i = i\frac{(b_i\ x_i\ -\ d_i) - (b_i\ x_{i+1}\ -\ d_{i+1})}{b_i\ -\ b_{i+1}} \tag{8}$$

for $i = 1, ..., m-1$, and

$$\alpha_m = m x_m, \tag{9}$$

and then showing that equations (4)-(6) hold. Equation (4) will be shown to hold by claim 2.1, equation (5) will be shown to hold by claim 2.3 and equation (6) will be shown to hold by claims 2.4 and 2.6.

Claim 2.1: $\alpha_i \geq 0,\ i = 1, ..., m$ and $\beta_i \geq 0,\ i = 1, ..., m-1$. Since (x,d) is envy free, person $i+1$ is not envious of person i: $b_{i+1}\ x_{i+1}\ -\ d_{i+1} \geq b_{i+1}\ x_i\ -\ d_i$. This then implies $(b_{i+1}\ x_{i+1}\ -\ d_{i+1}) - (b_{i+1}\ x_i\ -\ d_i) \geq 0$. Since $b_i\ -\ b_{i+1} > 0$, and i is a positive integer, that now means $\alpha_i = \frac{(b_{i+1}\ x_{i+1}\ -\ d_{i+1}) - (b_{i+1}\ x_i\ -\ d_i)}{b_i\ -\ b_{i+1}} i \geq 0$, for $i = 1, ..., m-1$. Since m is just a positive integer and $0 \leq x_m \leq 1$ is a portion of the object, so $\alpha_m = m x_m \geq 0$.

We also know person i is not envious of person $i+1$, so $b_i\ x_i\ -\ d_i \geq b_i\ x_{i+1}\ -\ d_{i+1}$, which implies $(b_i\ x_i\ -\ d) - (b_i\ x_{i+1}\ -\ d_{i+1}) \geq 0$, and therefore $\beta_i = \frac{(b_i\ x_i\ -\ d_i)-(b_i\ x_{i+1}\ -\ d_{i+1})}{b_i\ -\ b_{i+1}} i \geq 0$, for $i = 1, ..., m-1$.

Claim 2.2: $\alpha_i + \beta_i = i(x_i - x_{i+1})$ for $i = 1, ..., m-1$. This follows from the following algebra.

$$\begin{aligned}
\alpha_i + \beta_i &= i\frac{(b_{i+1}\ x_{i+1}\ -\ d_{i+1}) - (b_{i+1}\ x_i\ -\ d_i)}{b_i\ -\ b_{i+1}} + i\frac{(b_i\ x_i\ -\ d_i) - (b_i\ x_{i+1}\ -\ d_{i+1})}{b_i\ -\ b_{i+1}}, \\
&\quad \text{by equations (7) and (8)} \\
&= i\frac{(b_{i+1}\ x_{i+1}\ -\ d_{i+1}) - (b_{i+1}\ x_i\ -\ d_i) + (b_i\ x_i\ -\ d_i) - (b_i\ x_{i+1}\ -\ d_{i+1})}{b_i\ -\ b_{i+1}} \\
&= i\frac{(b_i - b_{i+1})(x_i - x_{i+1}) + d_i - d_i + d_{i+1} - d_{i+1}}{(b_i - b_{i+1})} \\
&= i(x_i - x_{i+1})
\end{aligned}$$

Claim 2.3: $\sum_{i=1}^{m} \alpha_i + \sum_{i=1}^{m-1} \beta_i = 1$ This follows from the following algebra.

$$\begin{aligned}
\sum_{i=1}^{m} \alpha_i + \sum_{i=1}^{m-1} \beta_i &= \sum_{i=1}^{m-1} (\alpha_i + \beta_i) + \alpha_m \\
&= \sum_{i=1}^{m-1} i(x_i - x_{i+1}) + m x_m, \text{ by claim 2.2 and equation (9)} \\
&= (x_1 - x_2) + 2(x_2 - x_3) + 3(x_3 - x_4) + ... + (m-1)(x_{m-1} - x_m) + m x_m \\
&= x_1 + (-x_2 + 2x_2) + (-2x_3 + 3x_3) + ... + (-(m-1)x_m + m x_m) \\
&= x_1 + x_2 + ... + x_m = 1
\end{aligned}$$

Claim 2.4: $x = \sum\limits_{i=1}^{m} \alpha_i y^i + \sum\limits_{i=1}^{m-1} \beta_i y^i$. Indeed, for $j = 1, ..., m$,

$$
\begin{aligned}
\sum_{i=1}^{m} \alpha_i y_j^i + \sum_{i=1}^{m-1} \beta_i y_j^i &= \sum_{i=1}^{m-1} (\alpha_i + \beta_i)\, y_j^i + \alpha_m y_j^m \\
&= \sum_{i=1}^{m-1} (i\,(x_i - x_{i+1})) y_j^i + m x_m y_j^m, \text{by Claim 2.2 and equation (9)} \\
&= \sum_{i=1}^{j-1} (i\,(x_i - x_{i+1})) y_j^i + \sum_{i=j}^{m-1} (i\,(x_i - x_{i+1})) y_j^i + m x_m y_j^m \\
&= \sum_{i=1}^{j-1} i\,(x_i - x_{i+1}) \cdot 0 + \sum_{i=j}^{m-1} i\,(x_i - x_{i+1})\,\frac{1}{i} + m x_m \frac{1}{m}, \text{ by equation (1)} \\
&= \sum_{i=j}^{m-1} (x_i - x_{i+1}) + x_m = x_j
\end{aligned}
$$

Claim 2.5: $\alpha_i b_i + \beta_i b_{i+1} = i(d_i - d_{i+1})$ for $i = 1, 2, \ldots, m-1$. This follows from the following algebra.

$$
\begin{aligned}
\alpha_i b_i + \beta_i b_{i+1} &= i\frac{(b_{i+1}x_{i+1} - d_{i+1}) - (b_{i+1}x_i - d_i)}{b_i - b_{i+1}} b_i + i\frac{(b_i x_i - d_i) - (b_i x_{i+1} - d_{i+1})}{b_i - b_{i+1}} b_{i+1}, \\
&\quad \text{by equations (7) and (8)} \\
&= i\frac{b_i b_{i+1} x_{i+1} - b_i d_{i+1} - b_i b_{i+1} x_i + b_i d_i + b_i b_{i+1} x_i - b_{i+1} d_i - b_i b_{i+1} x_{i+1} + b_{i+1} d_{i+1}}{b_i - b_{i+1}} \\
&= i\frac{b_i d_i - b_i d_{i+1} - b_{i+1} d_i + b_{i+1} d_{i+1}}{b_i - b_{i+1}} \\
&= i\frac{(b_i - b_{i+1})(d_i - d_{i+1})}{b_i - b_{i+1}} \\
&= i(d_i - d_{i+1})
\end{aligned}
$$

Claim 2.6: $d = \sum\limits_{i=1}^{m} \alpha_i e^i b_i + \sum\limits_{i=1}^{m-1} \beta_i e^i b_{i+1}$. Indeed for $j = 1, ..., m$,

$$\begin{aligned}
\sum_{i=1}^{m} \alpha_i e^i_j b_i + \sum_{i=1}^{m-1} \beta_i e^i_j b_{i+1} &= \sum_{i=1}^{m-1} e^i_j \left(\alpha_i b_i + \beta_i b_{i+1}\right) + e^m_j \alpha_m b_m \\
&= \sum_{i=1}^{m-1} e^i_j \left(i(d_i - d_{i+1})\right) + e^m_j \alpha_m b_m, \text{ by claim 2.5} \\
&= \sum_{i=1}^{j-1} (-\frac{1}{m}) \left(i(d_i - d_{i+1})\right) + \sum_{i=j}^{m-1} \left(\frac{1}{i} - \frac{1}{m}\right) \left(i(d_i - d_{i+1})\right) + 0 \cdot \alpha_m b_m, \\
&\quad \text{by equation (2)} \\
&= -\frac{1}{m} \sum_{i=1}^{m-1} \left(i(d_i - d_{i+1})\right) + \sum_{i=j}^{m-1} \left(\frac{1}{i}\right) \left(i(d_i - d_{i+1})\right) \\
&= -\frac{1}{m} \sum_{i=1}^{m-1} d_i + \frac{m-1}{m} d_m + d_j - d_m \\
&= -\frac{1}{m} \sum_{i=1}^{m} d_i + d_j = d_j
\end{aligned}$$

Section 3: We now show that the set of envy-free divisions, which we will denote as E, is the simplex determined by the vertices D. In the first two sections, we have already shown that E is the set of convex combinations of the divisions in D. Hence, the vertices of E is a subset of D. Since D contains $2m-1$ vectors, all of them will be vertices and E will be a simplex if we can show that E is $2m-2$ dimensional. We can do that by showing that subtracting one division in D from the $2m-2$ other divisions in D results in a linearly independent set. In particular, it is sufficient to show that $\{(y^n - y^m, b_n e^n);\ n = 1, ..., m-1\} \cup \{(y^n - y^m, b_{n+1} e^n);\ n = 1, ..., m-1\}$ is linearly independent.

To show linearly independence, we suppose

$$\sum_{i=1}^{m-1} \alpha_i \left(y^i - y^m, e^i b_i\right) + \sum_{i=1}^{m-1} \beta_i \left(y^i - y^m, e^i b_{i+1}\right) = (0, 0) \tag{10}$$

and will show that $\alpha_i = \beta_i = 0$ for all $i = 1, ..., m-1$. This will follow from claim 3.3:

Claim 3.1: $\alpha_i + \beta_i = 0$ for all $i = 2, ..., m$. Indeed, equation (10) is an equation involving a pair of m-vectors and the j-th component of the first vector is

$$\begin{aligned} 0 &= \sum_{i=1}^{m-1} \alpha_i \left(y_j^i - y_j^m\right) + \sum_{i=1}^{m-1} \beta_i \left(y_j^i - y_j^m\right) \\ &= \sum_{i=1}^{m-1} \left(\alpha_i + \beta_i\right) e_j^i, \text{ by equation (3)} \\ &= \sum_{i=1}^{j-1} \left(\alpha_i + \beta_i\right) e_j^i + \sum_{i=j}^{m-1} \left(\alpha_i + \beta_i\right) e_j^i \\ &= \sum_{i=1}^{j-1} \left(\alpha_i + \beta_i\right) (-\frac{1}{m}) + \sum_{i=j}^{m-1} \left(\alpha_i + \beta_i\right) (\frac{1}{i} - \frac{1}{m}), \text{ by equation (2)} \\ &= -\frac{1}{m} \sum_{i=1}^{m-1} \left(\alpha_i + \beta_i\right) + \sum_{i=j}^{m-1} \left(\alpha_i + \beta_i\right) \frac{1}{i} \end{aligned} \tag{11}$$

When $j = m$ equation (11) reduces to

$$0 = -\frac{1}{m} \sum_{i=1}^{m-1} \left(\alpha_i + \beta_i\right). \tag{12}$$

When $j = m - 1$ using equation (12), equation (11) reduces to

$$\begin{aligned} 0 &= -\frac{1}{m} \sum_{i=1}^{m-1} \left(\alpha_i + \beta_i\right) + \left(\alpha_{m-1} + \beta_{m-1}\right) \frac{1}{m-1} \\ &= \left(\alpha_{m-1} + \beta_{m-1}\right) \frac{1}{m-1} \end{aligned}$$

and thus we know $\alpha_{m-1} + \beta_{m-1} = 0$. Using the same process we can look at $j = m - 2$ and know that

$$\begin{aligned} 0 &= -\frac{1}{m} \sum_{i=1}^{m-1} \left(\alpha_i + \beta_i\right) + \left(\alpha_{m-2} + \beta_{m-2}\right) \frac{1}{m-2} + \left(\alpha_{m-1} + \beta_{m-1}\right) \frac{1}{m-1} \\ &= \left(\alpha_{m-2} + \beta_{m-2}\right) \frac{1}{m-2} + \left(\alpha_{m-1} + \beta_{m-1}\right) \frac{1}{m-1} \\ &= \left(\alpha_{m-2} + \beta_{m-2}\right) \frac{1}{m-2} \end{aligned}$$

and thus $\alpha_{m-2} + \beta_{m-2} = 0$. Continuing this process we can say $\alpha_i + \beta_i = 0$ for all $i = 2, ..., m-1$. We also know from equation (12) that $\alpha_1 + \beta_1 = 0$. It can be shown by

$$\begin{aligned} 0 &= -\frac{1}{m}\sum_{i=1}^{m-1}(\alpha_i + \beta_i) \\ &= -\frac{1}{m}(\alpha_1 + \beta_1) - \frac{1}{m}\sum_{i=2}^{m-1}(\alpha_i + \beta_i) \\ &= -\frac{1}{m}(\alpha_1 + \beta_1) \end{aligned}$$

thus $\alpha_1 + \beta_1 = 0$.

Claim 3.2: $\alpha_i b_i + \beta_i b_{i+1} = 0$ for all $i = 1, ..., m-1$. We can use equation (3) and the same procedure used in claim 3.1 to show when $\sum_{i=1}^{m-1} \alpha_i b_i e_j^i + \beta_i b_{i+1} e_j^i = 0$ we can say $\alpha_i b_i + \beta_i b_{i+1} = 0$ for all $i = 1, ..., m-1$. This is shown by realizing everywhere we see α_i we replace it with $\alpha_i b_i$ and everywhere we see β_i we replace it with $\beta_i b_{i+1}$, and the steps are the same.

Claim 3.3: $\alpha_i = \beta_i = 0$ for all $i = 1, ..., m-1$. Indeed suppose $i \in \{1, ..., m-1\}$.

From claim 3.1, $\alpha_i = -\beta_i$. By substituting this into claim 3.2 we obtain

$$\begin{aligned} 0 &= \beta_i b_{i+1} - \beta_i b_i \\ &= \beta_i (b_{i+1} - b_i) \end{aligned}$$

Since $b_{i+1} \neq b_i$, $\beta_i = 0$, thus by claim 3.2 $\alpha_i = 0$. □

The theorem makes the assumption that all valuations are different for each person. If some valuations are the same, then there is more than one labeling of the people that places the valuations in non-ascending order. This suggests that the corresponding set D could have many more vectors. For example, if $b_1 = b_2$, then giving the entire object to person 2 should be as possible as giving the entire object to person 1. On the other hand, people having equal valuations limits the possibilities for money transfers yielding envy-free divisions. Again for example, if $b_1 = b_2$ and the entire object is given to one of these two people, then there is only one way of transferring money among the people that will result in an envy-free division; hence, two vertices with different money transfers is replaced with a single vertex. For example in the toy car collection situation if Ann and Ben both thought the collection was worth \$24, and Carl thought the collection was worth \$12, then the initial five envy-free divisions would have changed to

Portion of Collection	Money Given
(Ann, Ben, Carl)	(Ann, Ben, Carl)
$\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$	$(0, 0, 0)$
$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$(2, 2, -4)$
$\left(\frac{1}{2}, \frac{1}{2}, 0\right)$	$(4, 4, -8)$
$(0, 1, 0)$	$(-8, 16, -8)$
$(1, 0, 0)$	$(16, -8, -8)$

the convex combinations of these divisions are still envy-free, notice that the average of the fourth and fifth divisions equals the third division. So, not all of the special divisions are vertices of the set of envy-free divisions. Thus, in this case the envy-free divisions are not a 5-dimensional simplex.

5. Conclusion

The modern treatment of fair division began with the work of Knaster as reported by Steinhaus [**5**]. One strand of the subsequent literature has focused on the cake-cutting problem: the fair division of a nonhomogeneous but finely divisible object with no money transfers allowed. For example, Brams and Taylor [**1**] found a procedure that finds an envy free division in a finite number of steps. Robertson and Webb [**3**] provide an introduction and overview of this strand of the literature. A second strand of the literature has focused on the fair division of indivisible objects facilitated by the transfer of money. For example, Meertens, Potters, and Reijnierse [**2**] showed under weak conditions an economy has an envy-free and Pareto efficient division.

In this paper, we have allowed both fine division of the object and money transfers. The focus has been on the simplest case: a single, homogeneous object and persons with additive valuations. By concentrating on the simplest case, we have characterized the entire set of envy-free solutions for any number of persons. It would be interesting to determine how this characterization generalizes to problems involving multiple objects, budget constraints, and/or more sophisticated valuation functions.

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